Your name and your lab partner(s): John and Joe Carmody **Section:** 6 **Date: 10-10-16**

Deriving a Mathematical Model for Variance in Bungee Spring Constant Dependent on Equilibrium Length

Abstract:

Given Hooke's law, our team's goal was to derive a mathematical model that described the relationship between the spring constant *k* and equilibrium length $L_{equilibrium}$ of a static displaced bungee cord system. To do so, we collected displacement measurements of different forces acting on a given bungee equilibrium length, and used them to estimate a spring constant of the bungee system for each of the five bungee equilibrium lengths. Overall, we collected six displacement measurements in response to varying forces for each equilibrium length to determine five different spring constants *k*. The results of all five restoring force vs. weight measurement data values were plotted individually. The slope of these graphs (*k*-constant) was then used to produce a spring constant *k* over bungee length L graph. Given the rate variance of *k*, we determined our model would be best described as a negative power function. Further tests seeking to validate this our model could choose a random equilibrium bungee length $L_{equilibrium}$ and a random $F_{restoringforce}$ and observe whether or not our determined equation predicts displacement within a reasonable range of static stretch events.

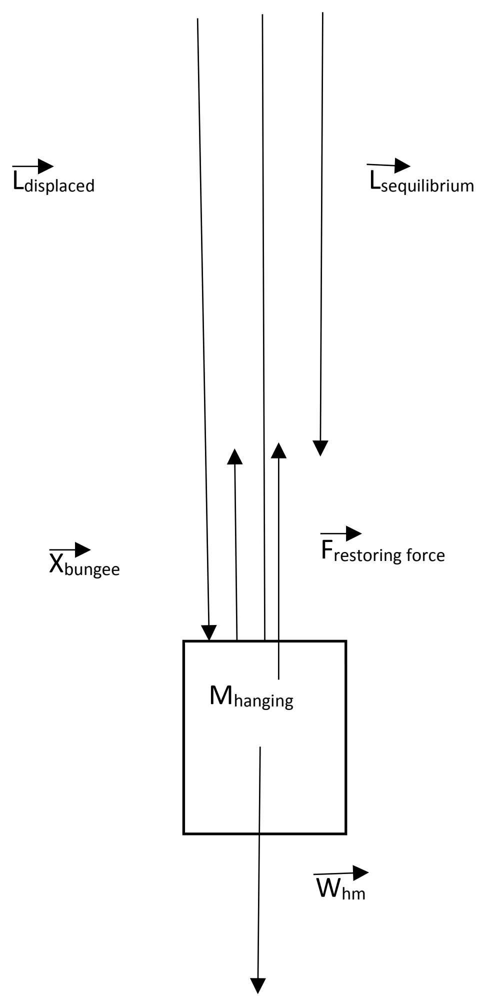
Introduction:

In this lab, our team's goal was to derive an appropriate spring-related model for bungee cords. In order to do so, we used Hooke's law to create a hypothetical model for a vertical bungee drop. This model was $F_{restoring\ force}=-k(L)x_{displacement}$, where k would vary dependent on the equilibrium length of the bungee. In doing so, we sought to determine the rate of variation in *k* as dependent on equilibrium length L. Our hypothesis was that as equilibrium length L increased, the restoring force constant *k* would decrease at an exponentially decreasing rate, this would ultimately be resolved by the rate law determination of our *k* over L plot.

Methods:

In our experimental setup, we determined to keep the equilibrium (undisplaced) length of the bungee cord constant for each one of our five equilibrium length values. In order to measure variance in spring constant *k* in response to length, we tested the displacement of the bungee in response to six forces for each of the five lengths.

Fig 1: A Free Body Diagram Depicting the Vectors Interacting with our Hooke's Law Vertical Bungee System. In order to derive a mathematical model for k($L_{equilibrium}$), we analyzed *k* at five different $L_{equilibrium}$ vector positions.



Eqtn. 1 Hypothetical Hooke's Law Adaptation for Determining the Variation in *k* with Respect to Equilibrium Length $L_{equilibrium}$. In order to solve for *k*(L), we isolated five *k* values at given lengths $L_{equilibrium}$. Given that $\vec{F}_{totsys}=0=\vec{F}_{restoringforce}+\vec{w}_{hm}$, we can algebraically set both sides equal to each other and take the scalar product of them, eliminating direction as a necessary component for subsequent data analysis.

$$\vec{F}_{restoringforce}=-k(\vec{L}_{equilibrium})\vec{x}_{bungee}$$

- Tied a length of our bungee cord to an overhang and measured its equilibrium length
- Tied this length of bungee to a hanger and measured the displacement of the bungee in response for six different weights
- Created a weight vs. displacement graph for the system

- Repeated steps one through three for the same bungee cable at five different equilibrium lengths
- Used slopes of weight vs. displacement graphs (more explicitly k value at a given equilibrium length) as points to graph spring constant vs. bungee length

Results:

In order to get the relevant result, the team measured and plotted restoring force $F_{restoringforce}$ over displacement $x_{displacement}$ for all five data sets and observed how k varied dependent on the equilibrium length of the bungee cable. k values were plotted against length, we then determined the most appropriate mathematical modeling system given its alignment with our data points.

Tables 1: Force vs. Displacement data for all five equilibrium bungee lengths. For each bungee length except .1030 m, we tested the same six $F_{restoringforce}$ values and measured displacement. Once again, we could neglect directionality of this unidirectional vector quantities due to algebraic adjustments and the scalar product of our hooke's law derivative, this eliminates the requirement of taking directionality into account for these next sections of data analysis. We estimated standard error of each variable based on our estimations of the relative accuracy of our tools for measurement.

Equilibrium (Unstretched) Length $L_{equilibrium}$ (m) (±0.001)	Spring Force due to Hanger System $F_{restoringforce}$ (N) (±.002)	Displacement $x_{displacement}$(m) (±0.0001)
0.103	0.196	0.007
0.103	0.294	0.012
0.103	0.392	0.016
0.103	0.491	0.021
0.103	0.589	0.025
0.103	0.688	0.031

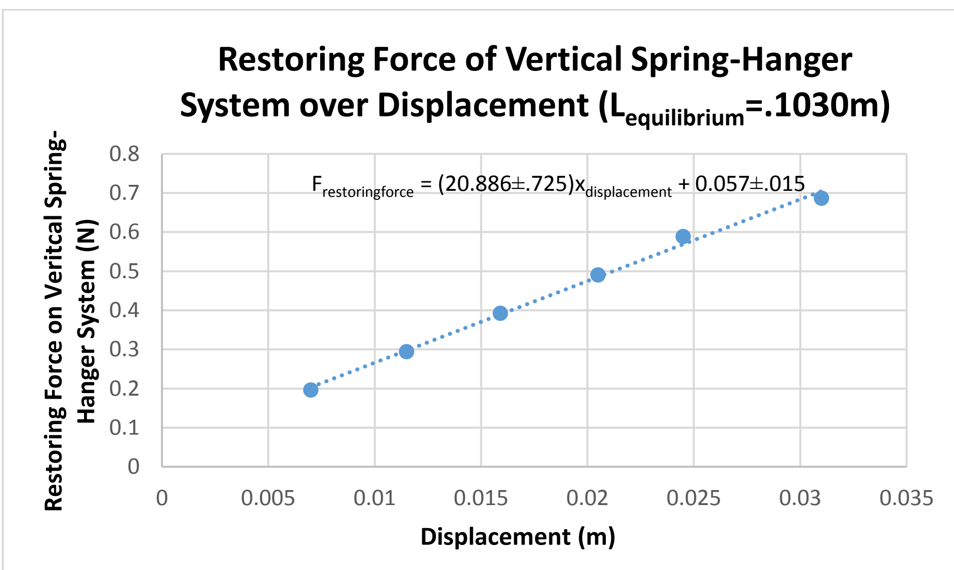
Equilibrium (Unstretched) Length $L_{equilibrium}$ (m) (±0.001)	Spring Force due to Hanger System $F_{restoringforce}$ (N) (±.002)	Displacement $x_{displacement}$(m) (±0.0001)
0.205	0.098	0.006
0.205	0.196	0.015
0.205	0.294	0.024
0.205	0.392	0.034
0.205	0.491	0.042
0.205	0.589	0.053

Equilibrium (Unstretched) Length $L_{equilibrium}$ (m) (±0.001)	Spring Force due to Hanger System $F_{restoringforce}$ (N) (±.002)	Displacement $x_{displacement}$(m) (±0.0001)
0.304	0.098	0.010
0.304	0.196	0.022
0.304	0.294	0.035
0.304	0.392	0.047
0.304	0.491	0.061
0.304	0.589	0.078

Equilibrium(Unstretched) Length $L_{equilibrium}$ (m) (±0.001)	Force due to Hanger System $F_{restoringforce}$ (N) (±.002)	Displacement $x_{displacement}$(m) (±0.0001)
0.399	0.098	0.014
0.399	0.196	0.027
0.399	0.294	0.042
0.399	0.392	0.061
0.399	0.491	0.080
0.399	0.589	0.101

Equilibrium (Unstretched) Length $L_{equilibrium}$ (m) (±0.001)	Force due to Hanger System $F_{restoringforce}$ (N) (±.002)	Displacement $x_{displacement}$(m) (±0.0001)
0.490	0.098	0.020
0.490	0.196	0.039
0.490	0.294	0.056
0.490	0.392	0.080
0.490	0.491	0.101
0.490	0.589	0.125

Graph 1: $F_{restoringforce}$ vs. x_{bungee} graphs for bungee equilibrium length .1030m. Where the slope of this graph represents the spring constant k of the bungee cord at the given equilibrium length.



Eqtn. 2 Hooke's law at equilibrium bungee length .1030 m. The k value of the system at this equilibrium length was 20.886±.725(N/m).

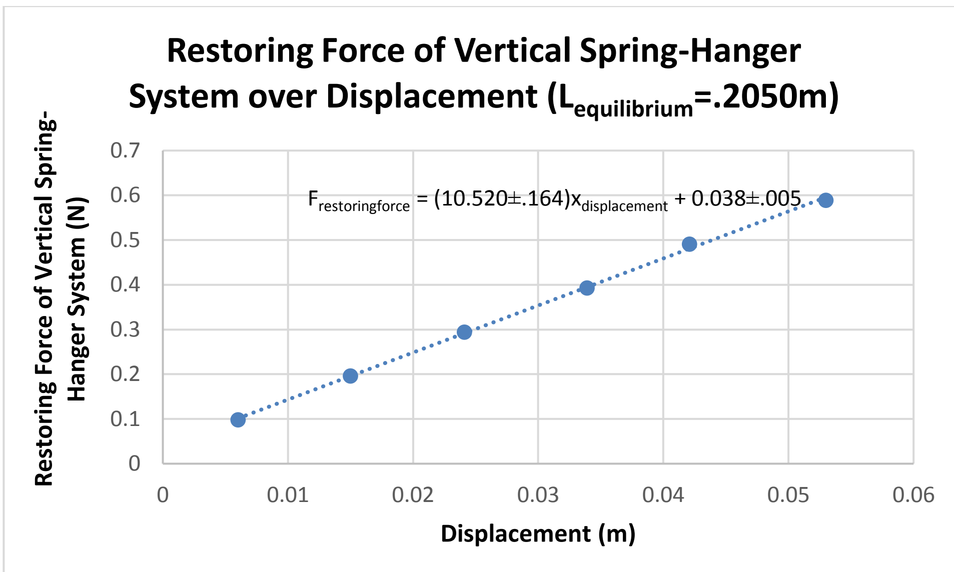
$$F_{restoringforce}=(20.886\pm.725)x_{displacement}+0.057\pm.015$$

uncertainty for slope=.725 % uncert=3.5%

uncertainty for y-intercept=.015 % uncert=26%

Given our data analysis, we determined spring constant k for the bungee whose equilibrium length $L_{equilirium}$=.1030m was 20.886±.725(N/m). We used Excel's linear regression analysis on our data points in order to calculate these values. Although our standard error for the y-intercept appeared large, the uncertainty in the y-intercept was very small making it less likely to seriously skew our data. Potential sources of this uncertainty could be due to measurements taken while the force of gravity was acting on the weight of the bungee cord itself at "equilibrium."

Graph 2: $F_{restoringforce}$ vs. x_{bungee} graphs for bungee equilibrium length .2050m. Where the slope of this graph represents the spring constant k of the bungee cord at the given equilibrium length.



Eqtn. 3 Hooke's law at equilibrium bungee length .2050m. The k value of the system at this equilibrium length was 10.520±.164(N/m).

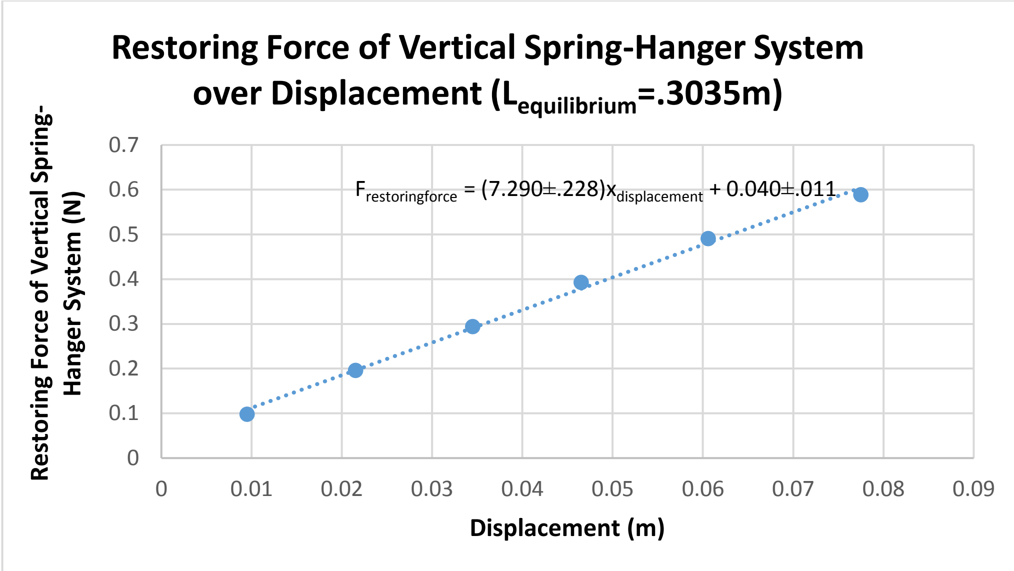
$$F_{restoringforce}=(10.520\pm.164)x_{displacement}+0.038\pm.005$$

uncertainty for slope=.164 % uncert=1.6%

uncertainty for y-intercept=.005 % uncert=13%

Given our data analysis, we determined the spring constant k for the bungee whose equilibrium length $L_{equilibrium}$=.2050m was 10.520±.164(N/m). Standard error was calculated using Excel's linear regression analysis on our data points. Although our standard error for the y-intercept appeared large, the uncertainty in the y-intercept was very small making it less likely to seriously skew our data. Potential sources of this uncertainty could be due to measurements taken while the force of gravity was acting on the weight of the bungee cord itself at "equilibrium."

Graph 3: $F_{restoringforce}$ vs. x_{bungee} graphs for bungee equilibrium length .3035m. Where the slope of this graph represents the spring constant k of the bungee cord at the given equilibrium length.



Eqtn 4. Hooke's law at equilibrium bungee length .3035m. The k value of the system at this equilibrium length was 7.290±.228(N/m).

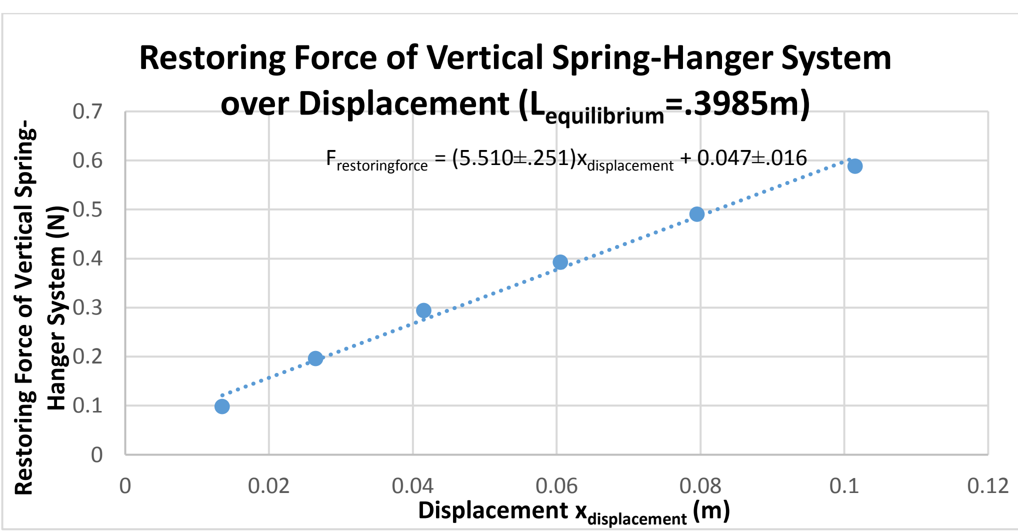
$$F_{restoringforce} = (7.290 \pm .228)x_{displacement} + 0.040 \pm .011$$

uncertainty for slope=.228 % uncert=3.1%

uncertainty for y-intercept=.011 % uncert=28%

Given our data analysis, we determined the spring constant k for the bungee whose equilibrium length $L_{equilibrium}$=.3035m was 7.290±.228(N/m). Standard error was calculated using Excel's linear regression analysis on our data points. Although our standard error for the y-intercept appeared large, the uncertainty in the y-intercept was very small making it less likely to seriously skew our data. Potential sources of this uncertainty could be due to measurements taken while the force of gravity was acting on the weight of the bungee cord itself at "equilibrium."

Graph 4: $F_{restoringforce}$ vs. x_{bungee} graphs for bungee equilibrium length .3985m. Where the slope of this graph represents the spring constant k of the bungee cord at the given equilibrium length.



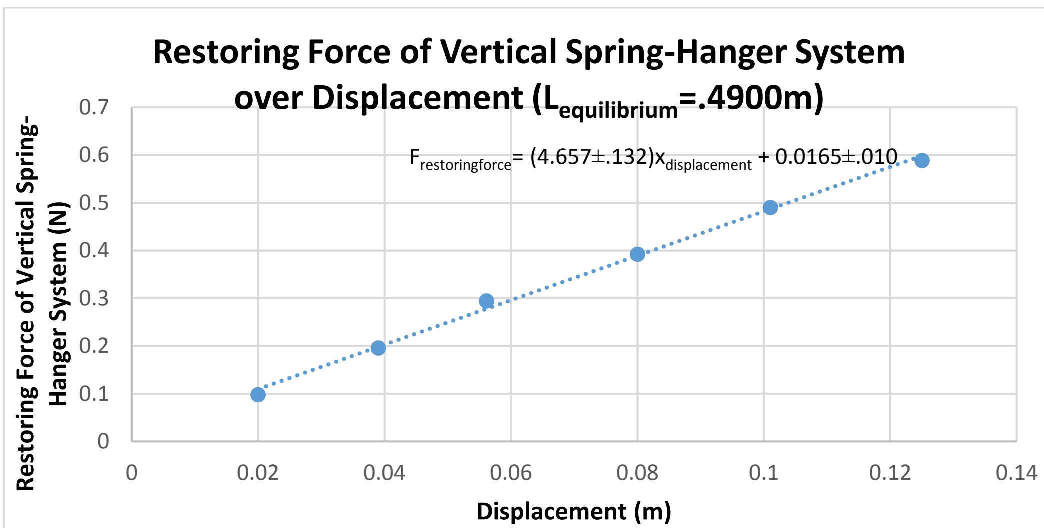
Eqtn 5. Hooke's law at equilibrium bungee length .3985m. The k value of the system at this equilibrium length was 5.510±.228(N/m).

uncertainty for slope=.251 % uncert=4.6%

uncertainty for y-intercept=.016 % uncert=34%

Given our data analysis, we determined the spring constant k for the bungee whose equilibrium length $L_{equilibrium}$=.3035m was 7.290±.228(N/m). Standard error was calculated using Excel's linear regression analysis on our data points. Although our standard error for the y-intercept appeared large, the uncertainty in the y-intercept was very small making it less likely to seriously skew our data. Potential sources of this uncertainty could be due to measurements taken while the force of gravity was acting on the weight of the bungee cord itself at "equilibrium."

Graph 5: $F_{restoringforce}$ vs. x_{bungee} graph for bungee equilibrium length .4900m. Where the slope of this graph represents the spring constant k of the bungee cord at the given equilibrium length.



Eqtn 5. Hooke's law at equilibrium bungee length .4900m. The k value of the system at this equilibrium length was 4.657±.132(N/m).

uncertainty for slope=.132 % uncert=2.8%

uncertainty for y-intercept=.010 % uncert=60%

Given our data analysis, we determined the spring constant k for the bungee whose equilibrium length $L_{equilibrium}$=.4900m was 4.657±.132(N/m). Standard error was calculated using Excel's linear regression analysis on our data points. Although our standard error for the y-intercept appeared large, the uncertainty in the y-intercept was very small making it less likely to seriously skew our data. Potential sources of this uncertainty could be due to measurements taken while the force of gravity was acting on the weight of the bungee cord itself at "equilibrium."

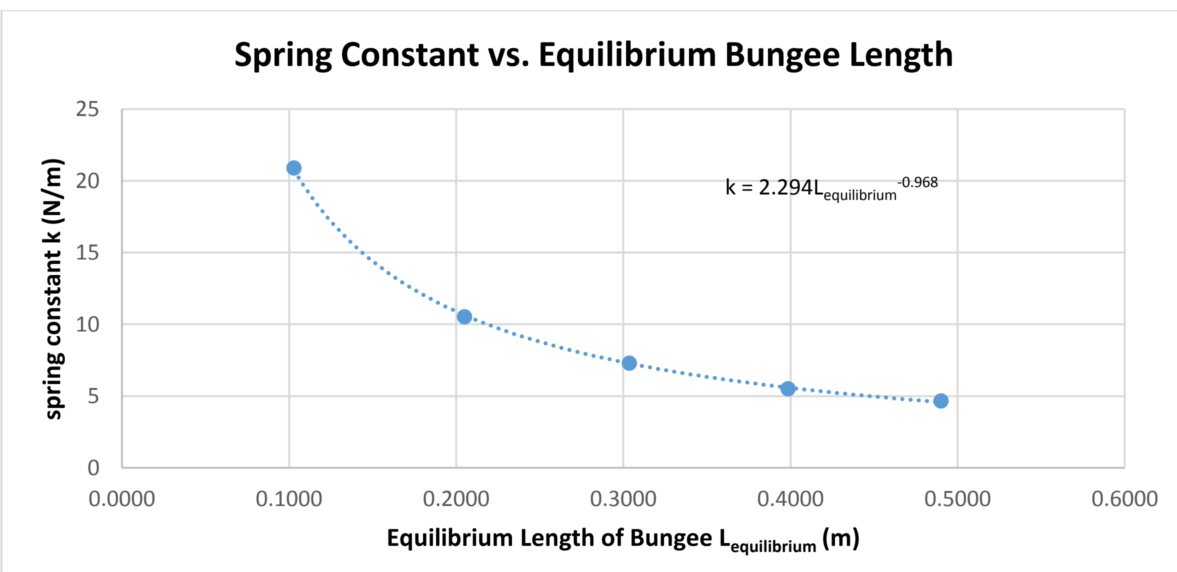
After determining the k values for each of these respective $L_{equilibrium}$ hanger-bungee systems, we then created a table that combined them all to give a quick visual data comparison between k-constants and bungee lengths. These data were graphed to determine the best rate variation relationship between spring constants for the bungee *k* and equilibrium bungee length $L_{equilibrium}$..

Table 2: calculated k value vs bungee length data table. k values were calculated using the slope of each one of the previous five spring force over displacement graphs.

k-constant **k** (N/m)	Bungee Length $\mathbf{L_{equilibrium}}$ (m)
20.89	0.1030
10.52	0.2050
7.29	0.3035
5.50	0.3985
4.66	0.4900

Given that our points exhibited a non-linear relationship, we had to use further analysis in order to determine the standard error for these data points.

Graph 6: k-constant over bungee length variance determination graph. In order to determine the relationship between the spring constant k and equilibrium bungee length $L_{equilibrium}$, we fit this graphical representation to the best fit model. In this case the model proved to be a power function



Eqtn. 6 Spring Constant vs. Equilibrium Bungee Length Graph for the Bungee-Hanger System under Observation. The variance between spring constant and Equilibrium Length of Bungee was best described using a negative power model.

$$\mathbf{k = 2.294L_{equilibrium}^{-0.968}}$$

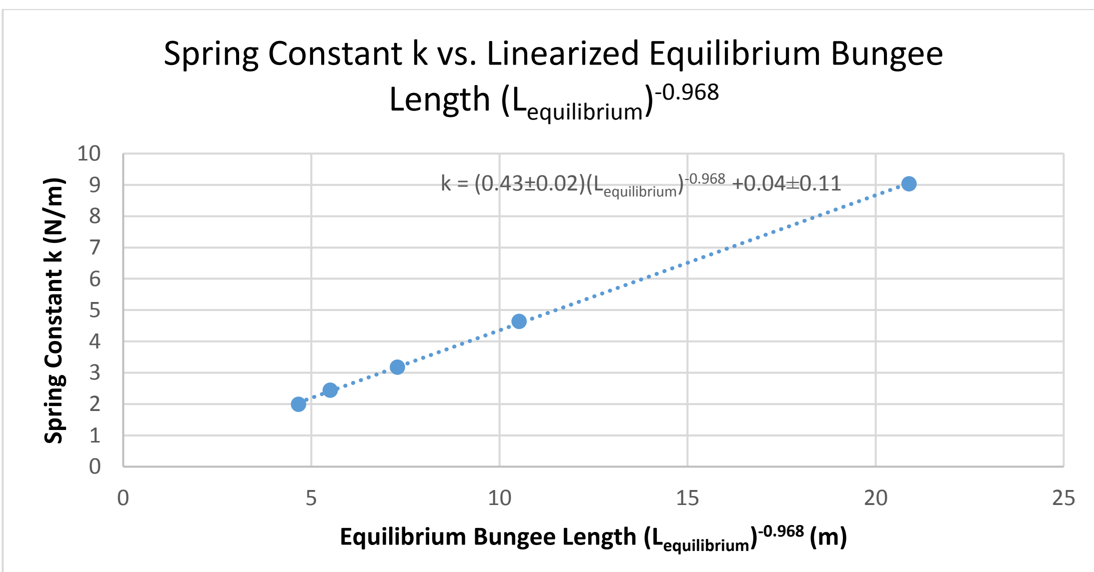
Despite the fact that we isolated a rough mathematical model which would describe the relationship between two essential variables for our system, we would be unable to identify respective standard error. In order to determine the standard error of the given data points, we had to linearize this data table and run regression analysis.

Table 3: "Linearized" Data Points Table. In this circumstance, all values of $L_{equilibrium}$ were taken to the power of -0.968, creating a dependent variable of $(L_{equilibrium})^{-0.968}$.

k-constant (N/m)	$(L_{equilibrium})^{-0.968} \pm 0.02$
20.89	9.03
10.52	4.64
7.29	3.17
5.50	2.44
4.66	1.99

Using a set of "linearized" data points, we then created a graph, subsequently we were able to use linear regression analysis in order to calculate the standard error of the data values.

Graph 7: "Linearized" k-constant over Bungee Length Variance Determination Graph. In order to linearize the previous graph, we transformed the equilibrium length values of $L_{equilibrium}$ on the x-axis to $(L_{equilibrium}{}^{-0.968})$



uncertainty for slope=0.02 % uncert=4.7%

uncertainty for y-intercept=0.11 % uncert=275%

Using our linearized dataset, we were able to identify a model for the relationship between our bungee-hanger system's equilibrium length and it's spring constant which took into account standard error. We calculated the standard error of this data set using linear regression analysis of our data points. Due to measurement errors described above, the percent uncertainty for our y-intercept proved astoundingly high with respect to our determined y-intercept. Given the purposes of our estimation (to estimate the relationship between *k* and $L_{equilibrium}$ at relatively much smaller bungee equilibrium lengths), the y-intercept value would prove to not be useful given that $L_{equilibrium}$ only approaches the y-intercept as $L_{equilibrium}$ approaches an infinite length. If left at a non-zero value within our equation, the error in the y-intercept would drastically skew our predictions for $L_{equilibrium}$ values not near infinity. Since zero is within the range of possible error values, we considered eliminating the influence of the uncertainty in the y-intercept.

Using our linearized graph, we are able to identify our definitive model for predicting the static displacement of a bungee system depending on its equilibrium length, this will hopefully guide us in our prediction of the motion of the egg in response to its relationship with the bungee during the egg drop. Located below is our final determined equation.

Eqtn. 7 Hooke's Law Variation Bungee-Hanger Model Prediction. This equation (similar to the one in our hypothesis section eqtn. 1) identified the rate of variance and standard error of the specific measurements we made.

$$\vec{F}_{restoringforce} = -(\vec{L}_{equilibrium}^{-0.968} \pm 0.02)\ \vec{x}_{displacement} + 0.04 \pm 0.11$$

Given this adaptation of Hooke's law, we can use our model to predict how the equilibrium length of a bungee will influence the motion of our bungee-egg drop system. As our results indicate:

$$k = L_{equilibrium}^{-0.968} \pm 0.02.$$

More broadly, our data determined that $F_{restoringforce} = (L_{equilibrium}^{-0.968} \pm 0.02) x_{displacement} + 0.04 \pm 0.11$. This equation will be important in modeling the bungee-egg drop event.

<u>Discussion:</u>

Through our experimental model, we identified a mathematical variation of Hooke's law which factored the equilibrium length of the bungee cord into the $F_{resorationforce}$ determination equation. In comparison to less nuanced models which lack this critical distinction ($k(L_{equilibrium})$ as opposed to just static k), our model exhibited greater accuracy in circumstances where $L_{equilibrium}$ varied. For instance, our equation directly predicts that for a bungee system of equilibrium length 0.3000m, the *k* value of this bungee system resided within the determined range of 3.19N/m and 3.23N/m. In a model without this distinction, a static calculated k value would be inadequate to describe motion of bungee systems without a constant $L_{equilibrium}$, given the serious limitations to this approach, it's hard to even envision what a comparable prediction would be without this nuanced distinction.

Uncertainty within our five $F_{restoringforce}$ over $x_{displacement}$ graphs included ten values. For the slope of these graphs, in the order in which they were introduced, standard errors of the slopes were equal to .725,.164,.228,.251, and .132. Percentage uncertainty of these values in order were 3.5%, 1.6%, 3.1%, 4.6%, and 2.8%. As far as the uncertainty in the y-intercept goes, it was generally drastically higher with respect to the y-intercept. The values of uncertainty for these graphs in order was .015, .005, .011, .016, and .010. The percentage uncertainty of these respective values was 26%, 13%, 28%, 34%, and 60%. Potential sources of error that could have contributed to the uncertainty built into our model include misreading tape measurements and potentially more drastic was failure to measure the $L_{equilibtium}$ value on a flat surface where the bungee cords weight would not contribute to its displacement. In the second component of our analysis, the respective calculated (using linear regression analysis) standard error values were 0.02 for the slope and 0.011 for the y-intercept. Percentage error values of the slope was 4.7% and y-intercept percentage error for this graphs was 275%. Given this drastic y-intercept uncertainty, we assumed this may have been partially due to $(L_{equilibrium})^{-0.968}$'s behavior as it approached zero (in this case it approached infinity).

Given the range of $L_{equilibrium}$ values we may potentially need for our bungee egg-drop event, it may be valuable to disregard the y-intercept of our linear equation given its abnormal amount of uncertainty and the way in which $L_{equilibrium}$ only approaches the y-intercept in conditions where $L_{equilibrium}$ nears infinity. In general, our main goal of this experiment (to determine $k(L_{equilibrium})$) appeared to succeed. In order to quantitatively test our model, one would have to predict a random set of $F_{restoringforce}$ and $L_{equilibrium}$ values, the accuracy of our model would be validated if the equation accurately predicted (within it's given error range) the value of $x_{displacement}$ given these conditions. Overall, we think our main results support our hypothesis, our original hypothesis professed that *k* would be dependent on $L_{equilibrium}$, a feature our data seems to support. Given we isolated this feature, our hypothesis predicted the existence of the relationship of the two. As far as the quantitative accuracy of our model goes, the only way of knowing for sure would be to test it, but in terms of accuracy of our hypothetical claims, our model matched our general, qualitative expectations.

Conclusion:

Using our independent calculations of *k* values from our static displacement measurements, then plotting their variation with respect to the equilibrium, we managed to derive a mathematical variation of Hooke's law that factored the equilibrium length of a bungee cord into the model. In a broader scope, this equation will help us determine our conditions for our bungee-egg drop event. Future experiments testing the validity of this model could test the displacement of a random equilibrium length in response to a random force and see whether or not the calculated value for displacement matches up to our equation within our determined error range.

On my honor, I have neither given nor received any unacknowledged aid on this assignment.

Pledged: John Carmody