Modeling the Spring Constant at Any Cord Length

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Abstract:

Our experiment is intended to find the general formula for spring constant of a cord at any length. Based on Hook's Law,

$$F = kx \; (1)$$

we can find the spring constant (k) at any fixed cord length by measuring the displacement to equilibrium (x) and the spring force (F). But the spring constant for one piece of the cord can't represent that of the entire cord. The spring constant varies as the cord length changes because of long molecule's property. Therefore, we measure the spring constants (k) at multiple initial lengths (l) and use regression analysis to find the relationship between k and l. First we take one fixed initial length, hang 10 different masses to the cord, and measure the displacement (x), investigating the spring constant (k) based on equation (1). Then we take 9 different cord lengths (l) and repeat the previous steps in order to find spring constants within a range of cord length. Plotting the spring constants (k) against respective cord length (l), we find k is inversely related to l. The linearization gives

$$k = 1.17l^{-1} \; (2)$$

The percent uncertainty is 2.7%, since it is quite small, so our result is relatively accurate. The source of uncertainty comes from the cord itself, since it extends spontaneously when the mass is added. The cord also presents quite different elasticity change under certain length range. Despite some error exceed uncertainty, our model matches data well at fairly large length, which would be our case of Bungee simulation.

Introduction:

Assuming that equation (1) applies to our cord, then at each initial length, the spring constant is fixed and the spring force (F) is proportional to displacement (x). However, the spring constant varies at different initial lengths. Thus, the relationship between the spring constant (k) and initial cord length (l) is crucial in order to model spring constant of our entire cord based on Hook's Law.

Hook's Law states that spring force (F) is proportional to displacement (x). In our spring-mass system, the net force at equilibrium position is zero according to Newton Second Law of Motion,

$$F_{net} = ma = F - W \ (3)$$

In our case, the spring force (F) equals the weight of hanging mass (W) and the latter is therefore proportional to displacement (x). The ratio will be the spring constant (k).

Yet one single case is not enough. As soon as we have spring constant (k) at multiple initial length (l) in a wild range, we can plot the function of $k = f(l)$, and predict k at any l. Our hypothesis is the k is inversely related to l.

Methods:

Hook's Law suggests that if the spring constant is the same, the displacement and the spring force change proportionally. So, for a certain cord length, we first hang 10 different masses to measure the displacement (x). Based on x and the weight of mass (W), we compute the spring constant at this certain length (l). Then we repeat the experiment for 9 different initial lengths, from each of which will give us a fixed spring constant. Finally, we use Excel to find the relationship between the spring constant and initial cord length.

We tie the plastic cord to the metal bar tightly so that it won't slide away. We also put a rule parallel to the cord for the convenience of measurement. The hanger (50g) is attached to a knot at the end of the cord. The whole system is anchored on a stable table. Our setup is shown in Figure 1.

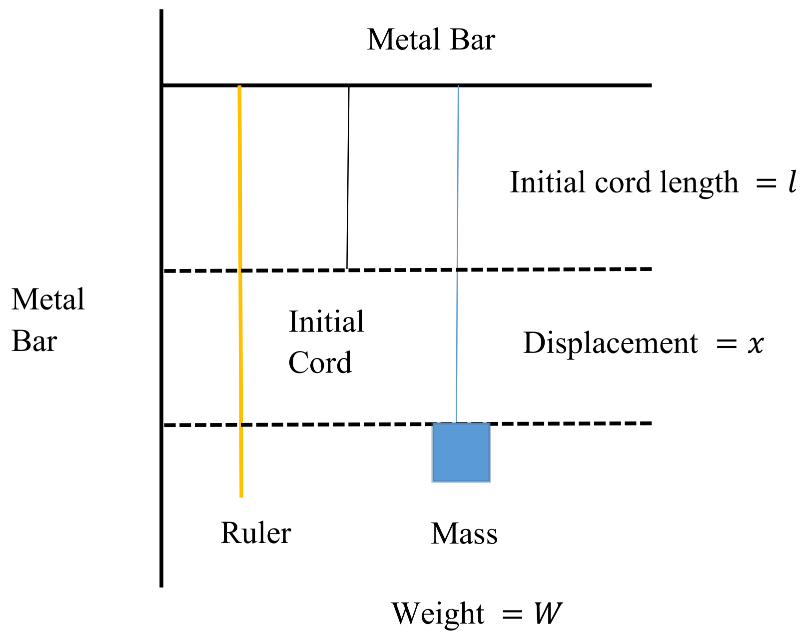


Figure 1: Sketch of setup. The cord and the ruler are attached to the metal bar and fall paralleled to each other, perpendicular to the level ground. The mass is added to the hanger at the end of the cord. The hanger is not shown here.

We conduct our experiment in the following procedures:

- We measure the initial cord length (l) at the knot.
- We add a 50g hanger to the cord and read the new equilibrium position at the knot.
- We add masses (up to 250 g, with an increment of 25g) to the hanger and read the new equilibrium positions at the knot.
- We calculate the displacements (x) by subtracting the initial cord length from the new equilibrium positions. We calculate the weight of each hanging mass (W).
- We calculate the spring constant
- We repeat the previous steps for ten different cord length (l) trails.

Results:

We measure the displacement (x) and the weights of hanging masses (W) for 9 trails. The data and the graphs for each trail are shown below, from chart 1 to 9 and figure 2 to 10. The follow trails are sorted on decreasing initial length. The summary on chart 10 is sorted in increasing initial length. (We actually do the experiment on random length order.)

Trail 1: Initial cord length is 0.65m.

Hanging Mass ($M \pm 0.01$kg)	Displacement ($x \pm 0.02$m)	Weight ($W \pm 0.01$N)
0.00	0.00	0.00
0.05	0.13	0.49
0.08	0.23	0.74
0.10	0.36	0.98
0.13	0.53	1.23
0.15	0.72	1.47
0.18	0.92	1.72
0.20	1.10	1.96
0.23	1.29	2.21
0.25	1.42	2.45

Chart 1: The displacement and weight of hanging masses under initial cord length 0.65m.

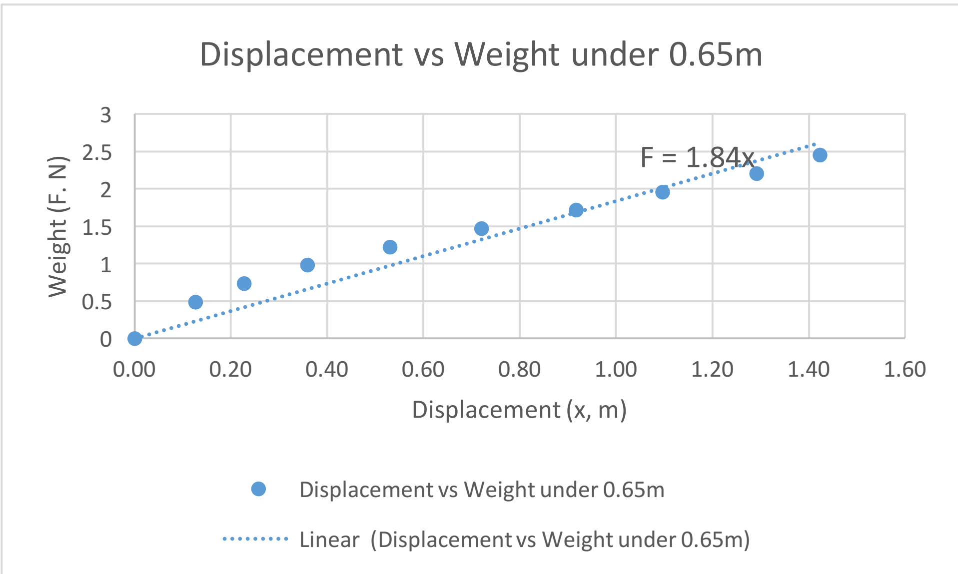


Figure 2: Linearization of displacement and weight based on Hook’s Law. The equation is F = 1.84x. The uncertainty is 4.4%f from regression analysis.

Trail 2: Initial cord length is 0.53m.

Hanging Mass ($M \pm 0.01$kg)	Displacement ($x \pm 0.02$m)	Weight ($W \pm 0.01$N)
0.00	0.00	0.00
0.05	0.11	0.49
0.08	0.17	0.74
0.10	0.30	0.98
0.13	0.44	1.23
0.15	0.61	1.47
0.18	0.78	1.72
0.20	0.91	1.96
0.23	1.07	2.21

Chart 2: The displacement and weight of hanging masses under initial cord length 0.53m.

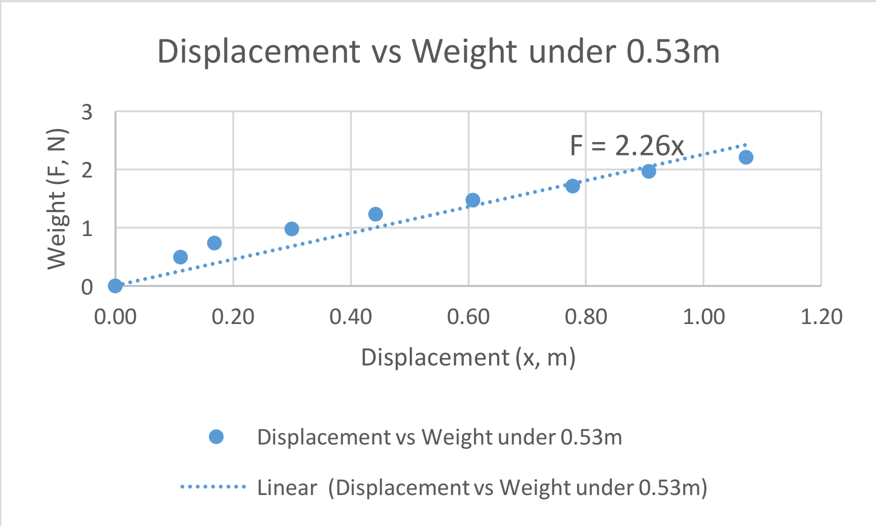


Figure 3: Linearization of displacement and weight based on Hook's Law. The equation is $F = 2.26x$. The uncertainty is 5.4% from regression analysis.

Trail 3: Initial cord length is 0.51m.

Hanging Mass ($M \pm 0.01$kg)	Displacement ($x \pm 0.02$m)	Weight ($W \pm 0.01$N)
0.00	0.00	0.00
0.05	0.12	0.49
0.08	0.20	0.74
0.10	0.28	0.98
0.13	0.43	1.23
0.15	0.58	1.47
0.18	0.75	1.72
0.20	1.07	1.96

Chart 3: The displacement and weight of hanging masses under initial cord length 0.51m.

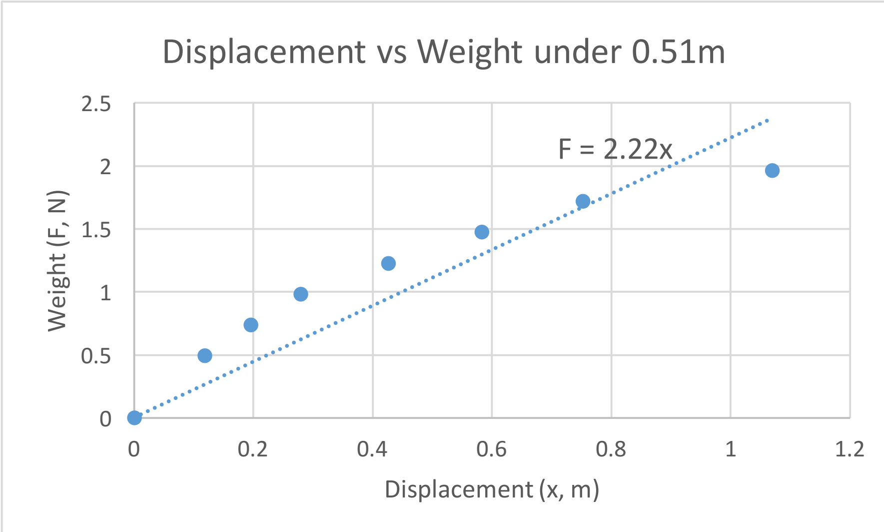


Figure 3: Linearization of displacement and weight based on Hook's Law. The equation is F = 2.22x. The uncertainty is 8.2% from regression analysis.

Trail 4: Initial cord Length is 0.46m.

Hanging Mass ($M \pm 0.01$kg)	Displacement ($x \pm 0.02$m)	Weight ($W \pm$ 0.01N)
0.00	0.00	0.00
0.05	0.13	0.49
0.08	0.17	0.74
0.10	0.30	0.98
0.13	0.41	1.23
0.15	0.55	1.47
0.18	0.73	1.72
0.20	0.88	1.96
0.23	1.03	2.21

Chart 4: The displacement and weight of hanging masses under initial cord length 0.46m.

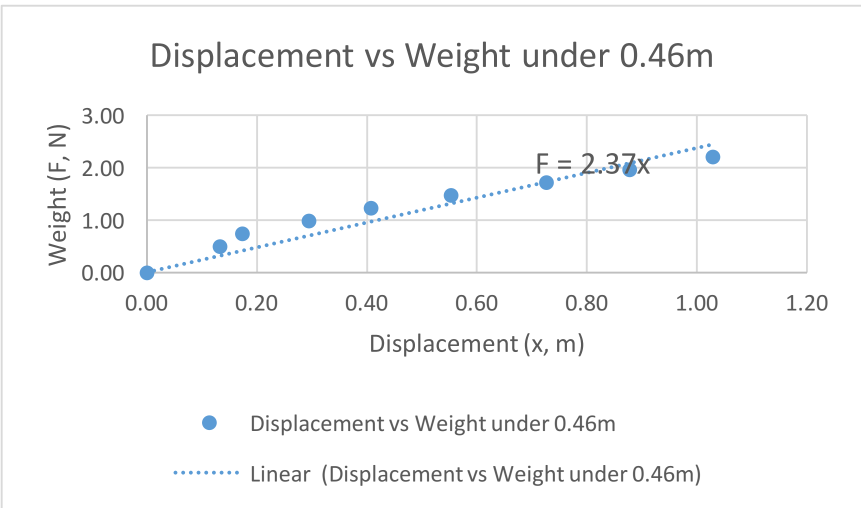


Figure 5: Linearization of displacement and weight based on Hook's Law. The equation is F = 2.37x. The uncertainty is 5.3% from regression analysis.

Trial 5: Initial cord length is 0.38.

Hanging Mass ($M \pm 0.01$kg)	Displacement ($x \pm 0.02$m)	Weight ($W \pm 0.01$N)
0.00	0.00	0.00
0.05	0.10	0.49
0.08	0.17	0.74
0.10	0.25	0.98
0.13	0.35	1.23
0.15	0.46	1.47
0.18	0.60	1.72
0.20	0.72	1.96
0.23	0.84	2.21
0.25	0.93	2.45

Chart 5: The displacement and weight of hanging masses under initial cord length 0.38m.

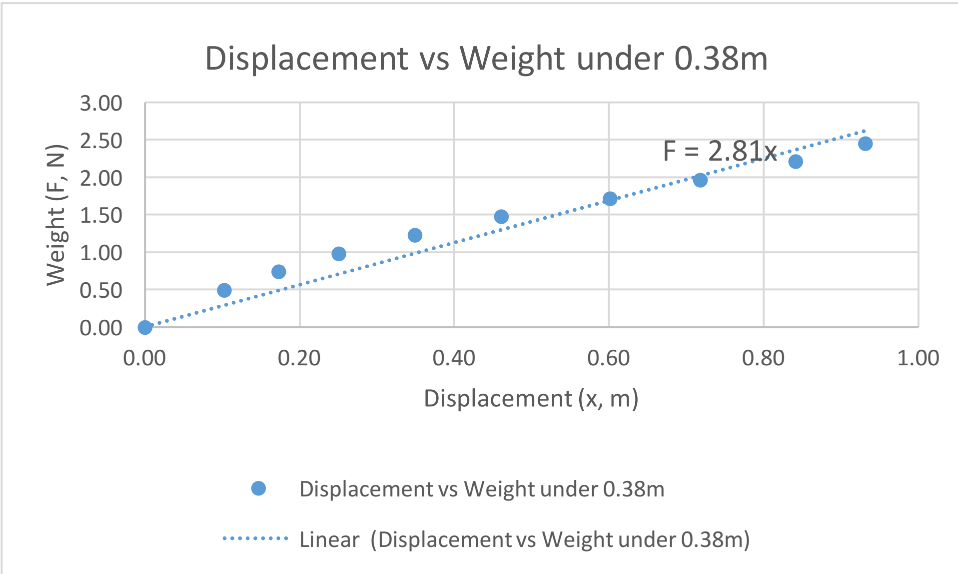


Figure 6: Linearization of displacement and weight based on Hook's Law. The equation is F = 2.81x. The uncertainty is 4.0% from regression analysis.

Trail 6: Initial cord length is 0.33m.

Hanging Mass ($M \pm 0.01$kg)	Displacement ($x \pm 0.02$m)	Weight ($W \pm 0.01$N)
0.00	0.00	0.00
0.05	0.07	0.49
0.08	0.12	0.74

0.10	0.21	0.98
0.13	0.28	1.23
0.15	0.45	1.47
0.18	0.60	1.72
0.20	0.69	1.96
0.23	0.77	2.21
0.25	0.88	2.45

Chart 6: The displacement and weight of hanging masses under initial cord length 0.38m.

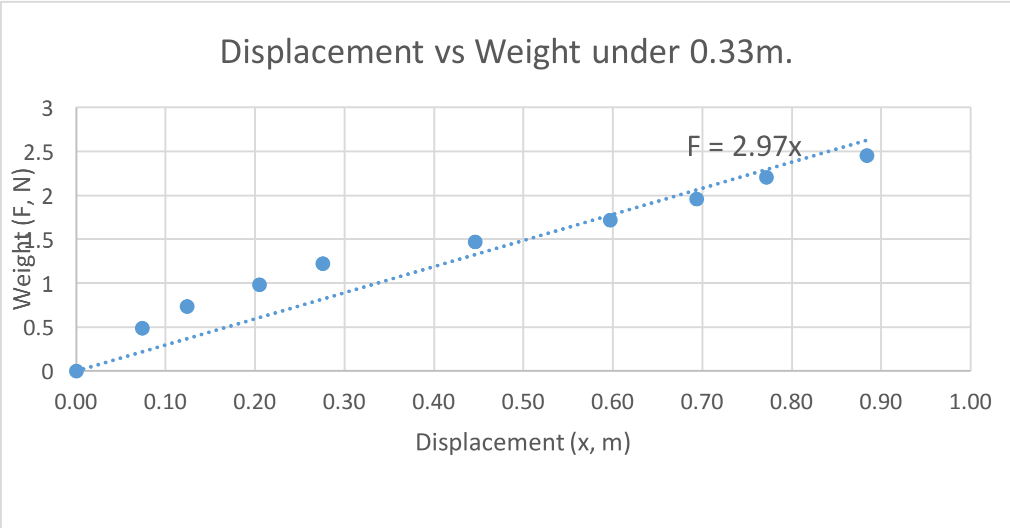


Figure 7: Linearization of displacement and weight based on Hook's Law. The equation is F = 2.97x. The uncertainty is 5.3% from regression analysis.

Trial 7: Initial cord length is 0.31m.

Hanging Mass ($M \pm 0.01$kg)	Displacement ($x \pm 0.02$m)	Weight ($W \pm 0.01$N)
0.00	0.00	0.00
0.05	0.08	0.49
0.08	0.15	0.74
0.10	0.25	0.98
0.13	0.36	1.23
0.15	0.49	1.47
0.18	0.64	1.72
0.20	0.79	1.96
0.23	0.95	2.21
0.25	1.06	2.45

Chart 7: The displacement and weight of hanging masses under initial cord length 0.31m.

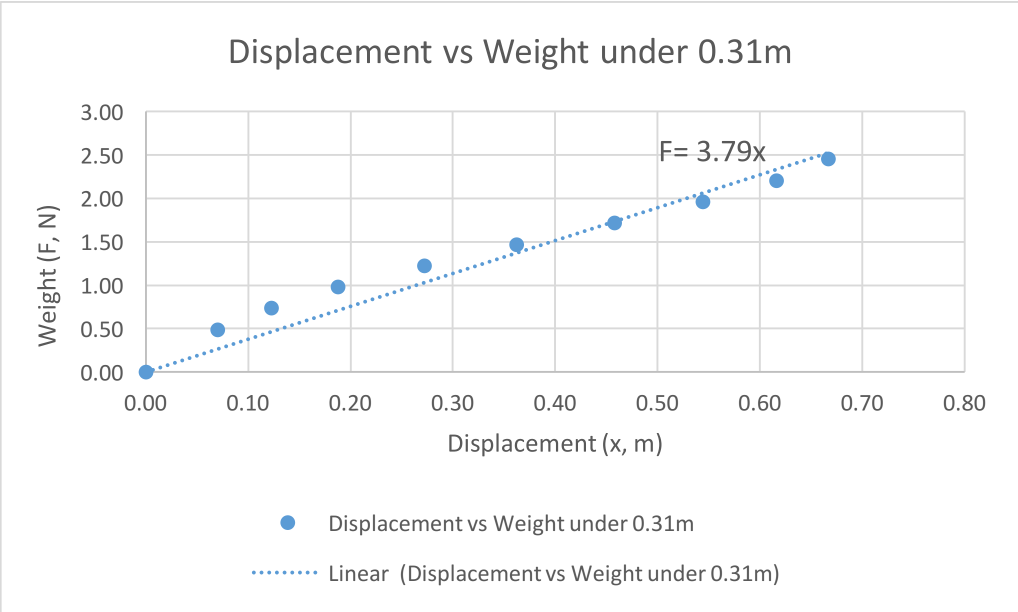


Figure 8: Linearization of displacement and weight based on Hook's Law. The equation is F = 3.79x. The uncertainty is 3.7% from regression analysis.

Trail 8: Initial length is 0.26m.

Hanging Mass ($M \pm 0.01$kg)	Displacement ($x \pm 0.02$m)	Weight ($W \pm 0.01$N)
0.00	0.00	0.00
0.05	0.05	0.49
0.08	0.09	0.74
0.10	0.13	0.98
0.13	0.20	1.23
0.15	0.28	1.47
0.18	0.35	1.72
0.20	0.42	1.96
0.23	0.49	2.21
0.25	0.56	2.45

Chart 8: The displacement and weight of hanging masses under initial cord length 0.26m.

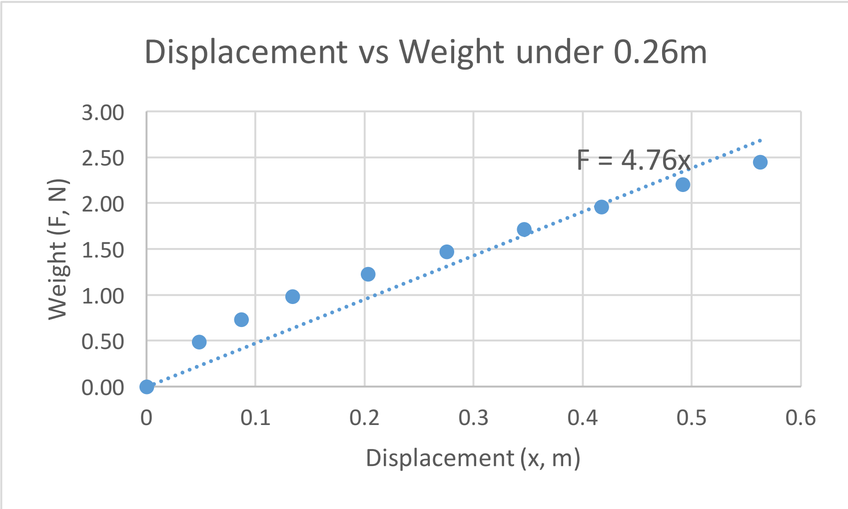


Figure 8: Linearization of displacement and weight based on Hook's Law. The equation is $F = 4.76x$. The uncertainty is 4.7 from regression analysis.

Trail 9: Initial cord length is 0.23m.

Hanging Mass ($M \pm 0.01$kg)	Displacement ($x \pm 0.02$m)	Weight ($W \pm 0.01$N)
0.00	0.00	0.00
0.05	0.05	0.49
0.08	0.08	0.74
0.10	0.12	0.98
0.13	0.16	1.23
0.15	0.23	1.47
0.18	0.30	1.72
0.20	0.37	1.96
0.23	0.43	2.21
0.25	0.49	2.45

Chart 9: The displacement and weight of hanging masses under initial cord length 0.23m.

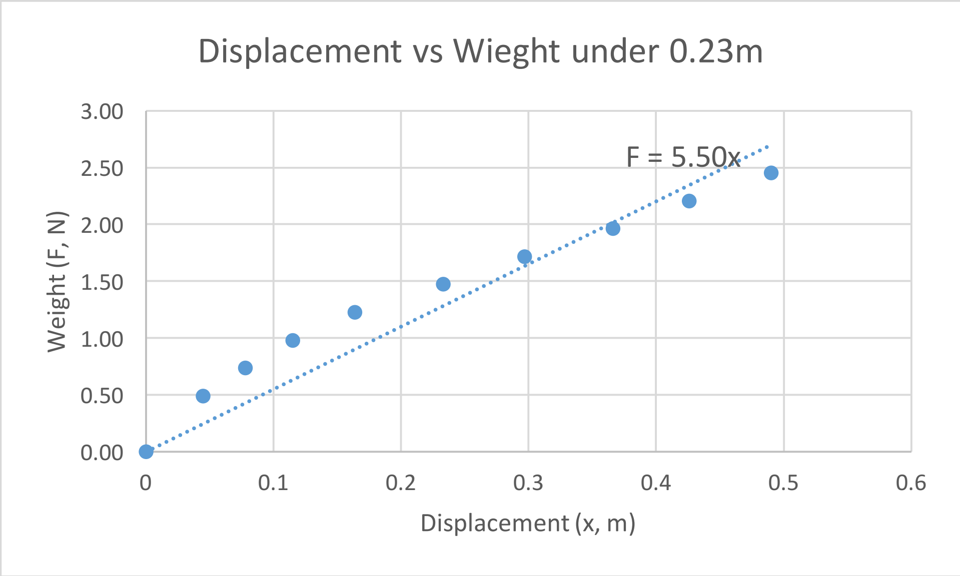


Figure 10: Linearization of displacement and weight based on Hook's Law. The equation is $F = 5.50x$. The uncertainty is 4.9% from regression analysis.

Based on Hook's Law, we know that the coefficient of cord length (x) in our linear equation for each trail is the spring constant (k). So now we know the spring constant for 9 trails, and are ready to conclude the relationship between the spring constant and the initial cord length. The spring constant (k) with respect to initial cord length (l) are shown in Chart 10. The power-fit is shown in Figure 11.

	Initial Length ($l \pm 0.01$m)	Spring Constant (k, $\mathrm{N} \cdot m^{-1}$)
Trial 1	0.23	5.50±4.9%
Trail 10	0.26	4.76±4.7%
Trial 2	0.31	3.79±3.7%
Trail 8	0.33	2.97±5.3%
Trial 6	0.38	2.81±4.0%
Trial 5	0.46	2.37±5.3%
Trail 9	0.51	2.22±8.2%
Trial 7	0.53	2.26±5.4%
Trial 3	0.65	1.84±4.4%

Chart 10: The spring constants at different initial cord lengths for all 9 trails. Trail 4 is not included because we don't tie the cord tightly and it slips.

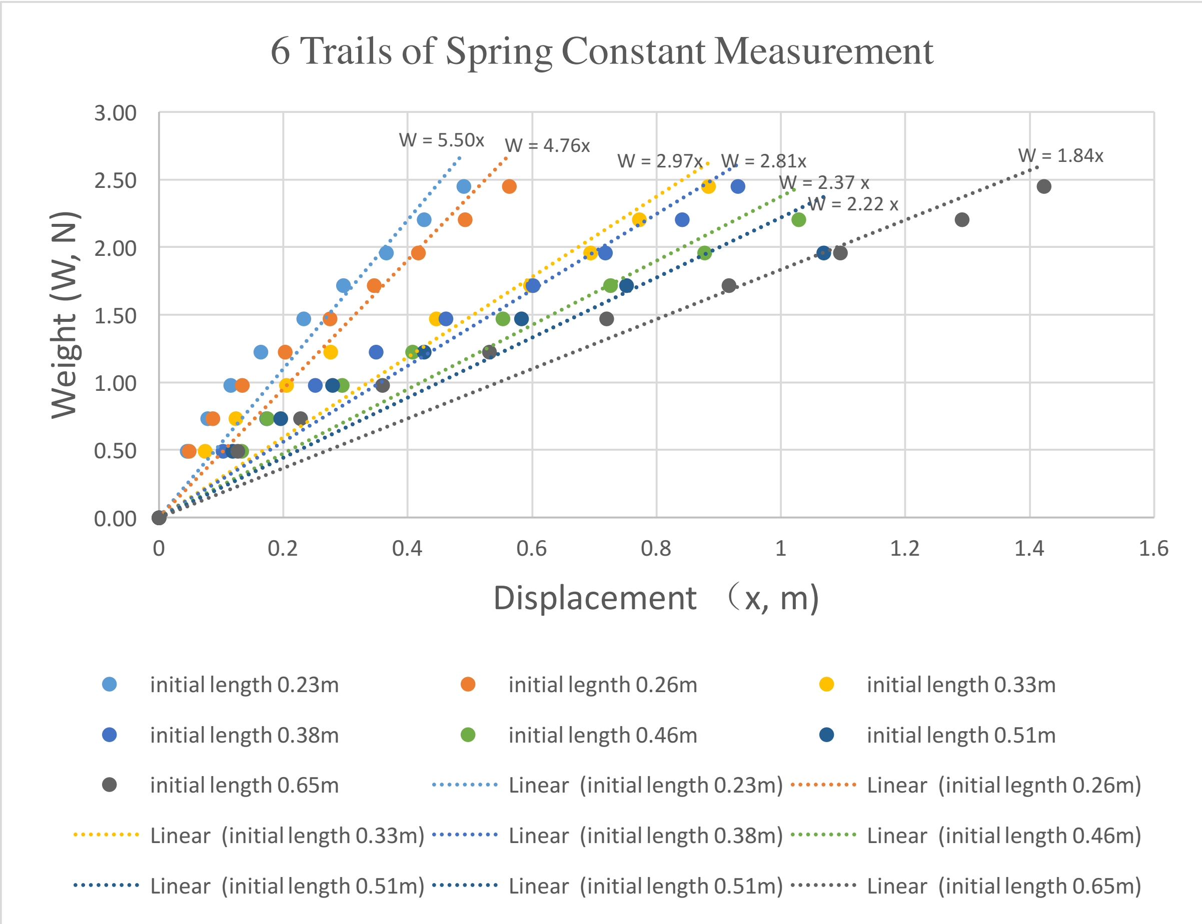


Figure 11: 6 selected trails' spring constants. We could see the inverse relationship between cord length and spring constant.

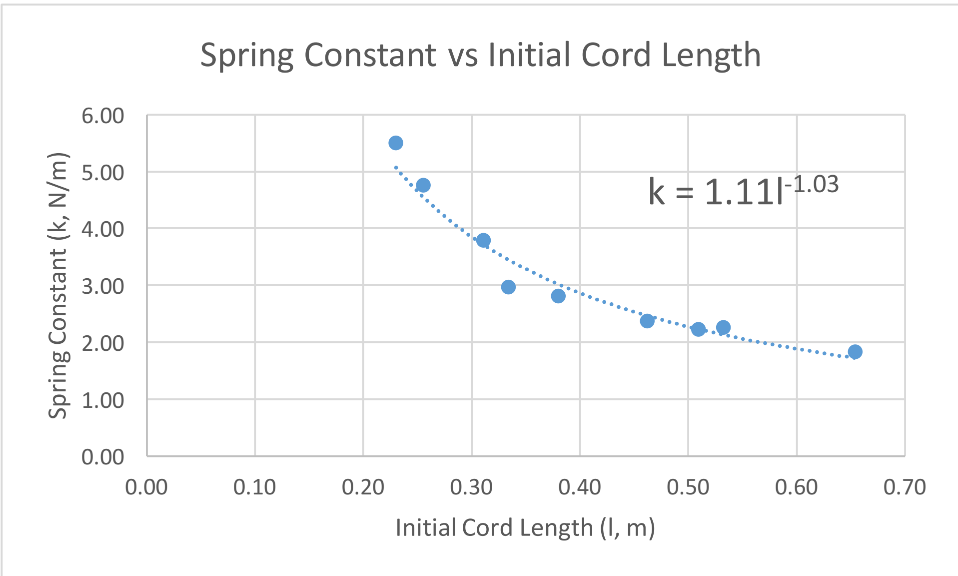
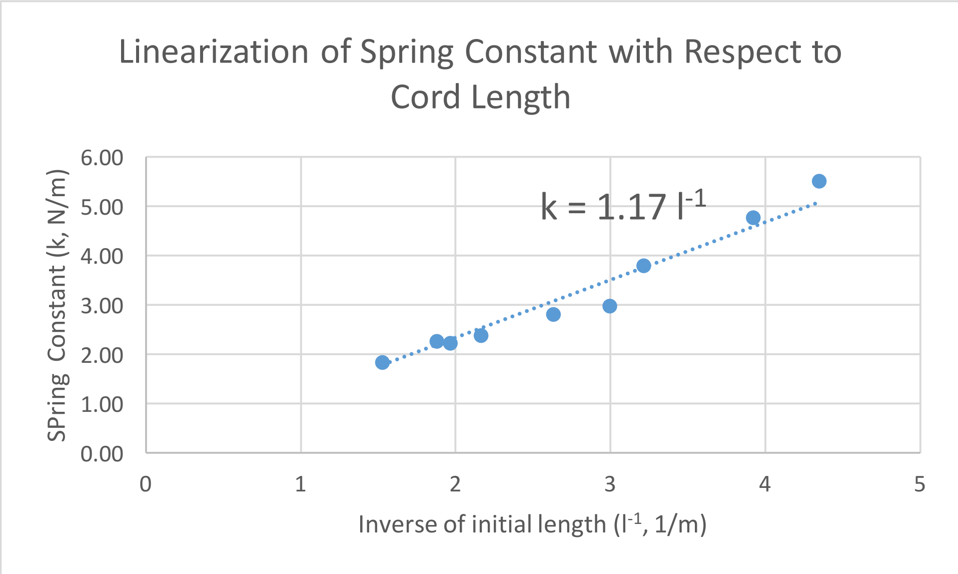


Figure 11: We plot the spring constants against the initial cord lengths. The power-fit equation is $k = 1.11l^{-1.03}$.

The linearization of spring constant (k) and the inverse of initial cord length (l^{-1}) is shown in Figure 12.



	Coefficients	Standard Error
Intercept	0	#N/A
X Variable 1	1.169515921	0.031267638

Firgure 12: Linearization of spring constant (k) over the inverse of initial cord length (l^{-1}). The equation is $k = 1.17l^{-1}$. Based on the Excel regression analysis, the uncertainty of spring constant is 0.03. The percent uncertainty is 2.7%.

In brief, we get 9 separate spring constants (k) under different initial lengths (l) calculated by Hook's Law. We plot these 9 pairs of k and l and apply regression analysis by Excel. The linearization gives us

$$k = 1.17l^{-1} \pm 2.7\% \quad (4)$$

which shows that spring constant is inversely related to initial cord length.

Discussion:

Based on equation (4), we calculate the theoretical values for spring constant (k) under different initial length (l). Then we compare the percent error with the percent uncertainty for each trail (see Chart 10). The result is shown in Chart 11.

	Initial Length	Real k Constant	Percent Uncertainty	Theoretical k Constant	Percent Error
Trial 1	0.23	5.50	4.9	5.09	7.6

Trail 10	0.26	4.76	4.7	4.59	3.7
Trial 2	0.31	3.79	3.7	3.76	0.7
Trail 8	0.33	2.97	5.3	3.50	19.4
Trial 6	0.38	2.81	4.0	3.08	9.7
Trial 5	0.46	2.37	5.3	2.53	6.6
Trail 9	0.51	2.22	8.2	2.30	3.4
Trial 7	0.53	2.26	5.4	2.20	2.2
Trial 3	0.65	1.84	4.4	1.79	2.5

Chart 11: Percent error and percent uncertainty for each trail. There are 4 trails (marked as red) in which percent errors is far larger than percent uncertainty.

Based on Chart 11, except for Trail 1 (0.23m), the percent error under extreme length value (super small or super large) is within the range of percent uncertainty. When it comes to middle length value, such as 0.33 (Trail 8), 0.38 (Trail 6), and 0.46 (Trail 5), the theoretical spring constant k value is more unstable. We conclude that the spring constant changes greatly under a normal stretched length (0.30-0.50m), but remains the same under extreme length (below 0.30m and above 0.50m).

The source of uncertainty comes from the elasticity of the cord. During each trail, we believe the cord adds some length to itself. The recorded cord length happened in Trail 7 and 8, shown below in Chart 12. For instance, in Trail 6, we measure the initial length is 0.33m. After adding masses to the cord, the cord ends up with 0.35m. Since the spring constant changes along with the cord length, it is reasonable to assume that the extra length gives additional uncertainty. Based on Chart 11, if we use the initial length of 0.33m, the percent error is 19.4%. We use the ending length of 0.35m, the percent error is 12.6%, the percent error dropped significantly. In Trail 7, the initial length is 0.53m, giving percent error of 2.6%. Yet the ended length is 0.55m, giving a percent error of 5.8%, which exceeds the percent uncertainty. Sadly, we did not notice this problem before the experiment. But first, this extended length adds more uncertainty; second, the extended length under 0.30m and 0.50m affects percent error in a different way, supporting our guessing that the cord present different elasticity pattern under different length range.

	Initial Length	Percent Uncertainty	Percent Error	Ending length	New Percent Error
Trial 7	0.53	5.4	2.2	0.55	5.8
Trail 8	0.33	5.3	19.4	0.53	12.6

Chart 12: The changed cord length in Trail 7 and 8. Based on the change, we recalculate the percent error. Oddly, one drops significantly while the other increases.

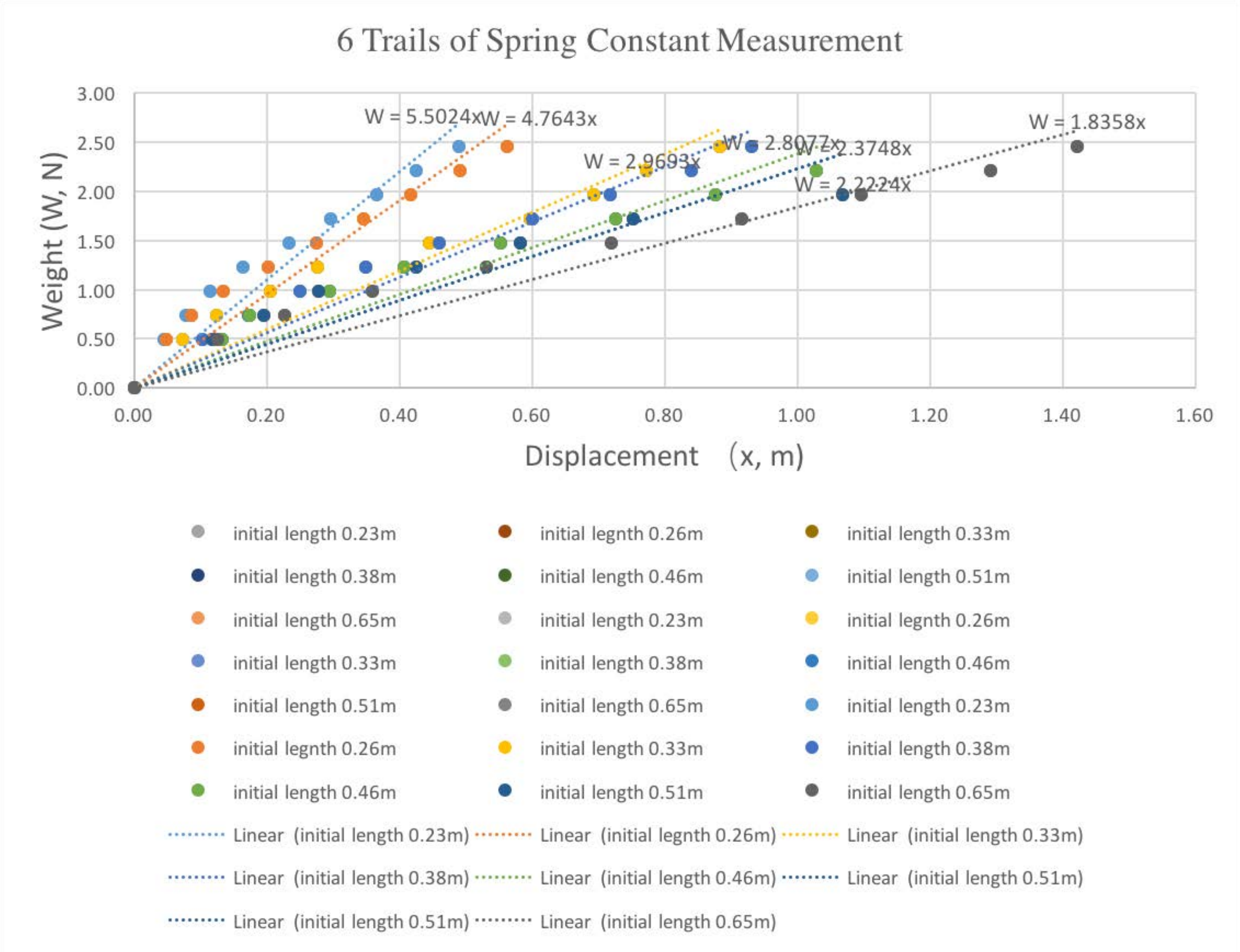
The other sources of uncertainty come from the reading the position scale. Our ruler is not completely parallel nor orthogonal to the ground, and our view is not parallel to the scale. The cord might not be unfolded completely even though we let it fall naturally. There is also different elasticity affecting spring constant for different section of our cord, given that it is unevenly starched out at first.

Overall, the linearization equation matches our hypothesis, that cord spring constant is inversely related to cord length. Although there are several large error, our model nevertheless matches data better especially under long cord length. Considering our Bungee jump will fully stretch the cord, our model is pretty useful. The only problem would be the ever-changing elasticity of the cord, making everything hard to predict.

CONCLUSION:

We examine the spring constant (k) at specific nine different cord lengths (l) in order to model the relationship between k and l. Based on Chart 10, linearization gives $k = 1.17l^{-1}$ with an uncertainty of 2.7%. The result matches our hypothesis that k is inversely related to l. Although our model has great error for length ranging from 0.30m to 0.50m, it gives relatively accurate prediction for length larger than 0.50m, which is useful for the Bungee simulation.

Our experiment presents some important aspects of the cord. First, as our model showed, the spring constant is inversely related to cord length. Second, the cord of different length range shows various elasticity property. Third, the cord may extend itself, adding more complexity on its elasticity. Those property helps us to understand and explain some seemingly weird results, while appreciating its enthralling complexity.



6 Trails of Spring Constant Measurement
Weight (W, N)
Displacement (x, m)
W = 5.5024x
W = 4.7643x
W = 2.9693x
W = 2.8077x
W = 2.3748x
W = 2.2224x
W = 1.8358x
3.00
2.50
2.00
1.50
1.00
0.50
0.00
0.00
0.20
0.40
0.60
0.80
1.00
1.20
1.40
1.60
initial length 0.23m
initial legnth 0.26m
initial length 0.33m
initial length 0.38m
initial length 0.46m
initial length 0.51m
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Linear (initial length 0.23m)
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Linear (initial length 0.65m)