Modelling Elastic Cord as an Ideal Spring:
The Relationship Between Spring Constant *k* and Equilibrium Length *l*

Kalady Osowski

ABSTRACT: For a given an elastic cord, there exists a relationship between the force placed on that cord and the distance it stretches. For this reason, it is common to model this behavior using Hooke's Law, where $\boldsymbol{F_{spring}} = -k\boldsymbol{x}$. However, the spring constant *k* of an elastic cord is dependent on the cord's length. The overarching purpose of this set of experiments was to model this relationship: to investigate how spring constant *k* changes as equilibrium length *l* of cord changes.

To determine this relationship, various masses were placed on a pre-tied length of cord, and the displacement of each mass was measured from the equilibrium length. Because the displacement was measured when the system was in equilibrium, it was known that the total force on the system was equal to zero. Since the only forces acting on the block were theoretically the weight of the mass and the spring force opposing it, the magnitude of spring force was taken to be equal to the magnitude of weight. This process of collecting equilibrium lengths, displacements, and spring force was repeated for several pre-tied lengths. Then, spring force was plotted versus displacement from equilibrium. Through this, the slope – equivalent to *k* – was determined for each equilibrium cord length. Finally, these individual *k* values were plotted against the equilibrium length at which they were found. This yielded an equation – and therefore a relationship - between the length of cord used and the resultant *k* value. When linearized, this relationship was found to be $k = 2.41\left(\frac{1}{l}\right) - \ 0.13$. This model allows one to predict *k* of the elastic cord given the length it is utilized at – information that can be used to predict forces of the elastic cord on the egg as a result of displacement during the Bungee Challenge.

INTRODUCTION:

Experimental Context:

This bungee experiment revolved around determining the relationship between *k*, the "spring constant" of an elastic cord modelled as an ideal spring, and the equilibrium length *l* of elastic cord used. This is essential for understanding inherent characteristics of the cord, which can be used to predict how the cord will react to an attached, falling mass in the Bungee Challenge.

Relevant equation(s) specific to this experimental purpose or setup:

$$\left|\boldsymbol{F_{spring}}\right| = m_{hanging}g = k|\boldsymbol{x}|$$

where

$\left|\boldsymbol{F_{spring}}\right|$ is the magnitude of the restoring force of the elastic cord, modelled by Hooke's Law as the spring force, which opposes the force of of the hanging mass; in Newtons

$m_{hanging}$ is the mass of the hanging mass; in kilograms

g is the gravitational constant; in meters/second^2

k is the spring constant of the cord at a specific length *l*; in Newtons/meter

$|\boldsymbol{x}|$ is the magnitude of displacement of the cord from its equilibrium position to its final position resulting from the mass hanging; in meters

$$|\boldsymbol{x}| = s - l$$

where

$|\boldsymbol{x}|$ is the magnitude of displacement of the cord from its equilibrium position to its final position resulting from the mass hanging; in meters

s is the stretched length of the elastic cord; in meters

l is the unstretched equilibrium length of the elastic cord; in meters

Basis or brief theoretical background:

Because the displacement was measured when the system was in equilibrium – where velocity is constantly 0 m/s – Newton's Second Law allows one to conclude that the total force on the system was equal to zero:

$$\boldsymbol{F_{total}} = ma = 0$$

Furthermore, the only forces acting on the block were theoretically the weight of the mass and the spring force opposing it. Therefore:

$$\boldsymbol{F_{total}} = m_{hanging}g - k\boldsymbol{x} = 0$$

From this, we can solve for $\boldsymbol{F_{spring}} = -k\boldsymbol{x}$ in terms of weight:

$$\boldsymbol{F_{spring}} = k\boldsymbol{x} = -m_{hanging}g$$

By this, one can see that while the direction of the forces are opposite, the magnitude of the weight is theoretically equal to the magnitude of the spring force. Therefore:

$$|\boldsymbol{F_{spring}}| = k|\boldsymbol{x}| = m_{hanging}g$$

The equation $|\boldsymbol{F_{spring}}| = m_{hanging}g = k|\boldsymbol{x}|$, in brief, shows that the restoring force is equal in magnitude to the force exerted by gravity due to the hanging mass. Using this relationship, one can graph $m_{hanging}g$ versus $|\boldsymbol{x}|$ for a specific equilibrium length of cord *l* to give a slope of *k* for that *l.*

Hypothesis: It seems that it takes more force to stretch a short length of cord to a certain displacement than a longer cord to that same displacement. Therefore, as the length of the cord increases, it is predicted that *k* should decrease.

METHODS:

Overall method and its rationale:

In brief, various masses were hung from a pre-tied length of cord. The value of $m_{hanging}g$ was calculated for each mass, and the displacement $|\boldsymbol{x}|$ of each mass was measured. The $m_{hanging}g$ values were then plotted against magnitude of displacement $|\boldsymbol{x}|$. This is because we know that

$$m_{hanging}g = |\boldsymbol{F}_{spring}|$$

so

$$m_{hanging}g/|\boldsymbol{x}| = |\boldsymbol{F}_{spring}|/|\boldsymbol{x}|$$

By Hooke's Law,

$$|\boldsymbol{F}_{spring}|/|\boldsymbol{x}| = k$$

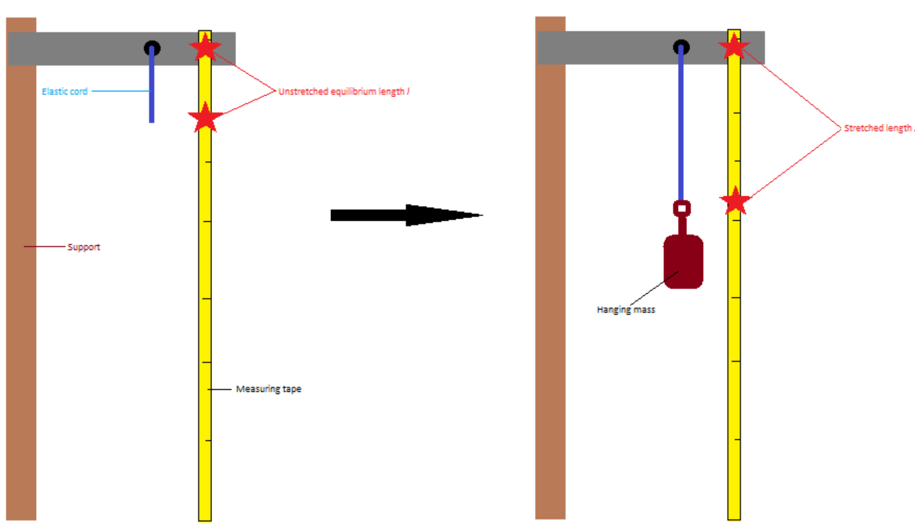
Therefore, the slope of the graph $m_{hanging}g$ versus $|\boldsymbol{x}|$ is k, the spring constant for that pre-tied length of cord.

This mass-hanging and displacement-measuring process was repeated with the same masses at several pre-tied lengths to find each k for a certain equilibrium length l of cord. Finally, a graph of k versus equilibrium length of cord l was created in order to determine the mathematical relationship between those two variables. It is this resultant equation that can be used later in the bungee challenge in order to determine k given a certain length of cord – and therefore the magnitude of the restoring force $|\boldsymbol{F}_{spring}|$ that a certain length of cord will apply on the egg.

Setup:

A support is erected vertically. An elastic cord is tied off into two small loops – one at the top, one at the bottom - and hung from the support. The unstretched length of this cord l is measured via a measuring tape attached to the support. A hanging mass is hung on the bottom loop. The new stretched length of the cord s is measured via the attached measuring tape. The setup is illustrated below.

Figure 1: Equilibrium cord length and stretched cord length. *A pre-tied length of cord* l *is measured. A mass is hung on the cord, and the cord's stretched length* s *is re-measured. The difference in the stretched length* s *and pre-tied length* l *represents the magnitude of displacement* $|\boldsymbol{x}|$.



Procedure:

1. A support was erected. A measuring tape was attached to the top of the support.
2. The elastic cord and masses weighing 5, 15, 25, 30, 40, and 50 grams were collected.
3. The cord was tied off with two small loops.
4. The cord was hung on the support, and the measuring tape was used to determine the unstretched length *l.* Note that *l* is the distance from the top of the first loop to the bottom of the other loop. The exact equilibrium lengths *l* used in this experiment were 0.230, 0.285, 0.425, 0.810, 1.005, and 1.18 meters.
5. Starting with the smallest 0.230 meters, the 5 gram mass was hung on the cord. The cord stretched to a new length *s.* This new length *s* was recorded*.* This same process was repeated on the 0.230 meter cord using 15, 25, 30, 40, and 50 grams.
6. Using this data, a graph of $|\boldsymbol{F_{spring}}|$ (calculated as $m_{hanging}g$) versus displacement $|\boldsymbol{x}| = s - l$ was created. The slope of this graph was determined to be *k* for the equilibrium length 0.230 meters.
7. Step 5 was repeated using each equilibrium length 0.285 meters. 0.425, 0.810, 1.005, and 1.18 meters.
8. Step 6 – building a graph of force versus displacement for each equilibrium length above and taking its slope – was repeated for each equilibrium length. At the end of the process, the *k* value corresponding to each equilibrium length was known.
9. To determine the relationship between *k* and equilibrium length *l, t*he above equilibrium lengths 0.230, 0.285, 0.425, 0.810, 1.005, and 1.18 meters were plotted against their corresponding *k* values to create the graph *k* vs. *l.* This graph showed an equation of $k = 2.337\ l^{-1.003}$.
10. The above graph was linearized by plotting *k* vs. 1/*l.* This yielded a relationship $k = 2.414\ (1/l) - 0.134$.
11. A regression analysis was performed on the linearized data.

RESULTS:

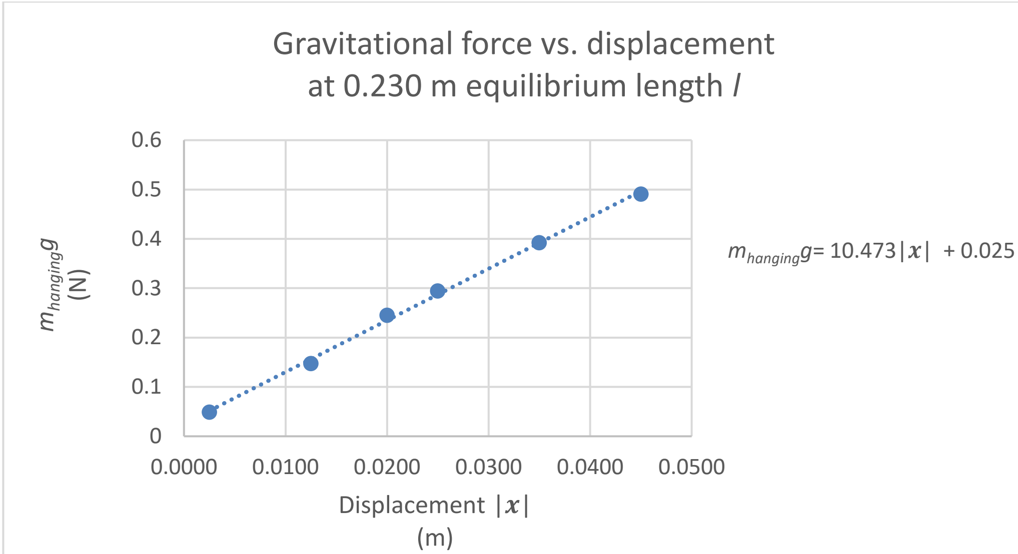
To begin, data including equilibrium cord lengths *l,* stretched cord lengths *s,* and $|\boldsymbol{F_{spring}}| = m_{hanging}g$ values were collected given a certain equilibrium cord length.

Figure 2: Force vs. displacement table for 0.230 meter equilibrium length. *Below is an example of one of seven data tables produced, each corresponding to a different equilibrium length of cord. Using this information, graphs of gravitational force versus displacement were produced at each equilibrium length.*

Hanging Mass $m_{hanging}$ (kg) (± 0.0001 kg)	**Gravitational Force** $m_{hanging}g$ (N) (± 0.001 N)	**Displacement** $\lvert\boldsymbol{x}\rvert$ (m) (± 0.002 m)
0.005	0.0491	0.0025
0.015	0.147	0.0125
0.025	0.245	0.0200
0.030	0.294	0.0250
0.040	0.392	0.0350
0.050	0.491	0.0450

Using the data from tables like the example in Figure 2, a graph of $m_{hanging}g$ versus $|\boldsymbol{x}| = s - l$ was created for each equilibrium cord length *l*. Since $m_{hanging}g = |\boldsymbol{F_{spring}}|$, Hooke's Law was manipulated to interpret the slope. It was found that the slope of each of these graphs $|\boldsymbol{F_{spring}}|/|\boldsymbol{x}|$ was theoretically *k* for that particular equilibrium length *l.* In total, there were seven such graphs produced, which gave the *k* values of the elastic cord when its equilibrium length was 0.230 meters, 0.285 meters. 0.425, 0.630 meters, 0.810, 1.005, and 1.18 meters.

Figure 3: Graph of gravitational force versus displacement for a cord of equilibrium length 0.230 meters. *Since* $m_{hanging}g = |\boldsymbol{F_{spring}}|$*, the slope of this graph is equal to* $|\boldsymbol{F_{spring}}|/|\boldsymbol{x}|$*, or* k *by Hooke's Law.*



Equation: $m_{hanging}g = |\boldsymbol{F_{spring}}| = 10.473|\boldsymbol{x}| + 0.0254$

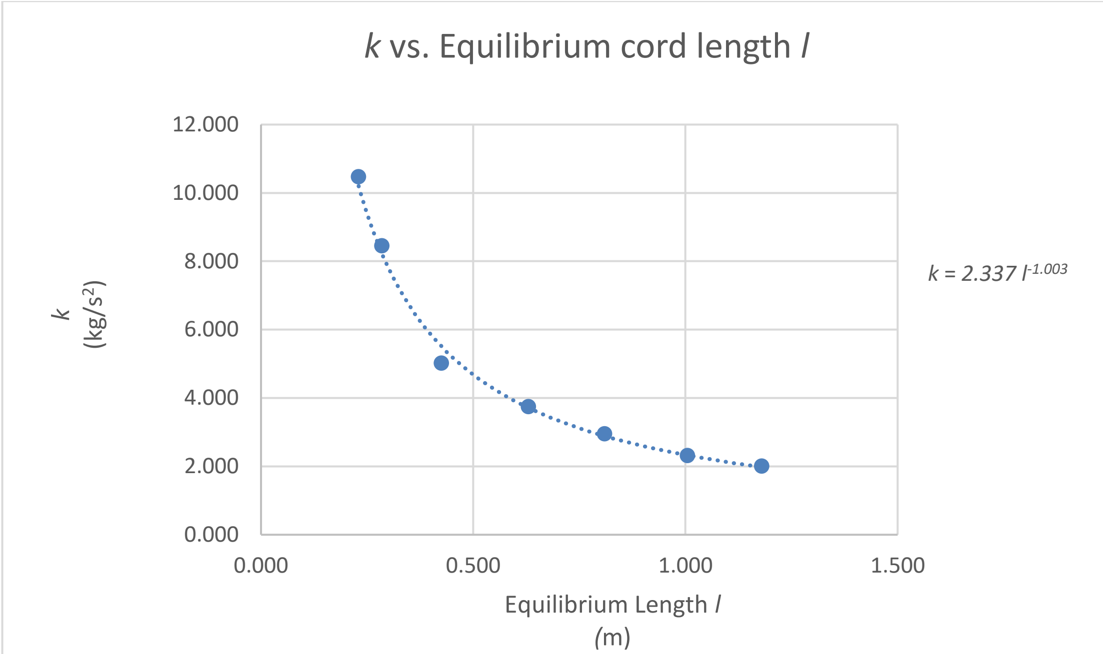
The *k* values from each of the seven individual Gravitational Force vs. Displacement graphs were collected along with their corresponding equilibrium lengths and put into a second chart Figure 4.

Figure 4: Calculated spring constant at various equilibrium lengths. *From the Gravitational Force versus Displacement graphs (modelled by Figure 3), the slope for each equilibrium length of cord was collected. This value is theoretically* k*, the spring constant in Hooke's Law.*

Equilibrium Length *l* (m) (± 0.002 m)	**Spring Constant** *k* (kg/s^2)
0.230	10.473
0.285	8.458
0.425	5.016
0.630	3.751
0.810	2.950
1.005	2.317
1.180	2.004

From the data in Figure 4, new graph was created. This plotted *k* versus equilibrium length *l* to determine their relationship. The resultant equation from the linearized graph *k* vs. 1/*l* provided the relationship between equilibrium length of the cord and its *k* value at that length.

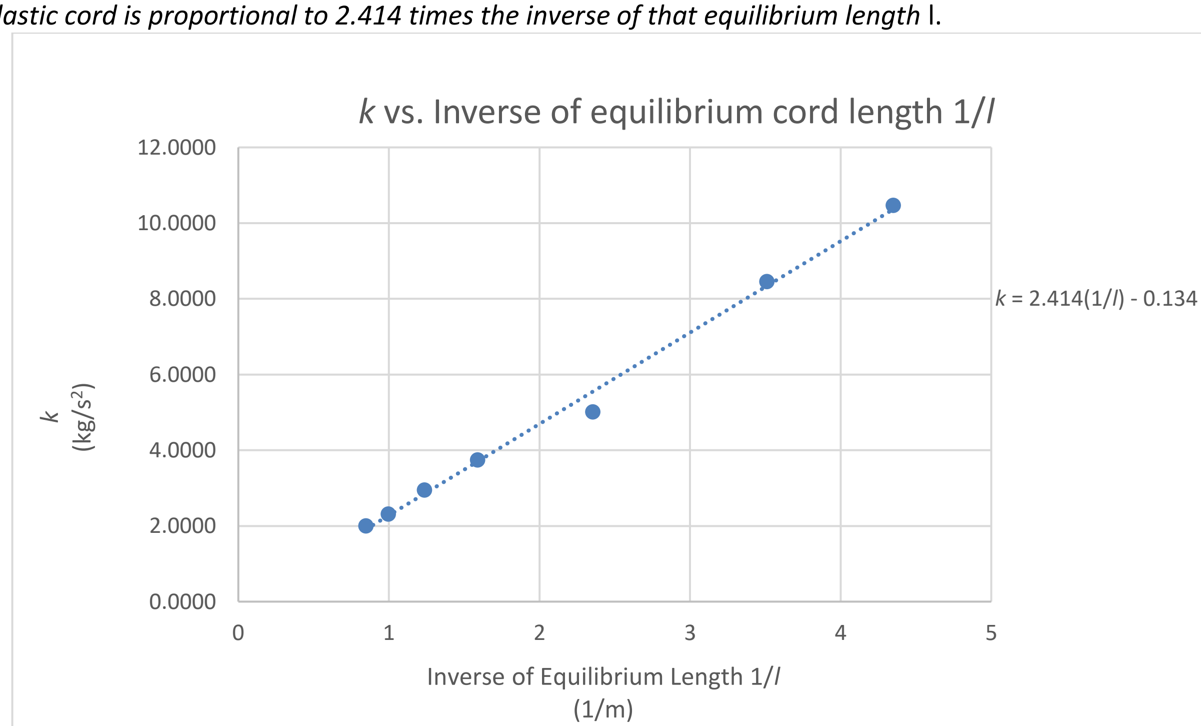
Figure 5: k *versus Equilibrium cord length* l*. The spring constant of a certain equilibrium length* l *of elastic cord is proportional to 2.337 times approximately the inverse of that equilibrium length* l.



Equation: $k = 2.337\ l^{-1.003}$

A power function was fit to the curve shown in Figure 5. To linearize it, the values of k were plotted against $1/l$, where l was the equilibrium length of the cord corresponding to that k. The resultant graph provided the mathematical relationship between k and l that was originally sought.

Figure 6: k *versus Inverse of equilibrium cord length 1/*l*. The spring constant of a certain equilibrium length* l *of elastic cord is proportional to 2.414 times the inverse of that equilibrium length* l.



Linear equation: $k = 2.414(1/l) - 0.134$

An uncertainty analysis was performed on Figure 6. Below are the results of that analysis.

Use *Excel* regression analysis on any graph that has a **linear** fit only (see EG), to obtain:

uncertainty for slope= 0.078 % uncert= 3%

uncertainty for y-intercept= 0.192 % uncert= 143%

Identify experimental value(s) of interest:

The experimental value of interest is the slope of the *k* vs. Inverse of equilibrium cord length 1/*l* graph because this value is what relates the equilibrium length of elastic cord to the spring constant at that length. This equation allows one to find *k* given some length of cord.

value obtained = 2.414

uncertainty of experimental value(s) = 0.078 % uncert= 3%

name the technique used for propagation of uncertainty: Excel Regression Analysis

Summarize Results:

By plotting $m_{hanging}g$ versus displacement, *k* for a certain equilibrium length of cord was determined. Then, *k* was plotted against *l* to determine their relationship. The linearized relationship of *k* and *l* was found to be described as $k = 2.414(1/l) - 0.134$. The uncertainty of the slope of this equation was found to be 0.078, yielding 3% uncertainty in that value.

DISCUSSION:

The overarching purpose of this set of experiments was to model this relationship: to investigate how spring constant *k* changes as equilibrium length *l* of cord changes. To test this, values $m_{hanging}g$ was plotted displacement at a given equilibrium length. From this, *k* for a certain equilibrium length *l* of cord was determined. Then, *k* was plotted against *l* to determine their relationship. The linearized relationship of *k* and *l* was found to be described as $k = 2.414(1/l) - 0.134$.

Using *Excel Regression* Analysis, the uncertainty of the slope of this equation was found to be 0.078. Given the experimental slope value of 2.414, this yields 3% uncertainty in that value. This allows one to conclude that this calculated slope value is precise – that is, the values used to create it conformed to the linear trend closely. However, because a model of how this specific cord's *k* value changes with respect to *l* does not exist, an error analysis cannot be completed. However, to test the accuracy of this model, it would be appropriate to pick an equilibrium length l_{test} within the range of those used to build the model – 0.500 meters, for instance – and to run the first six steps of the original procedure. This would result in a Gravitational Force vs. Displacement graph for a new equilibrium length l_{test}, and the slope k_{graph} could then be retrieved. Then, we could calculate a theoretical k_{model} using the equation $k_{model} = 2.414(1/l_{test}) - 0.134$. Through a comparison of k_{graph} and k_{model}, one could have a better understanding of how accurately this mathematical model predicts the observable results. Overall, to better the model, it would be beneficial to test a wider range of masses on a wider range of equilibrium lengths.

Sources of uncertainty:

The main source of uncertainty in the experiment certainly comes from measuring the equilibrium length *l,* the stretched length *s,* and therefore the magnitude of displacement $|x|$ of the elastic cord. First, when the cord was tied, two loops had to be made in the cord in order to hang it and hang a mass from it. The size of these loops, while small, varied in between experiments, and the amount these loops stretched differed from how the rest of the cord stretched. Second, it was difficult at times to measure *l*, as the unweighted cord would curl up and one had to pull it straight, trying to prevent the application of additional pressure. Lastly, in order for the hanging masses to not hit the measuring tape, the cord was a small distance from the measuring tape, so reading the lengths was not as precise as it could have been. In brief, these three elements affect the integrity of the value of $|\boldsymbol{x}|$, which directly affects equilibrium length *l*, the value *k* determined at each equilibrium length *l,* and ultimately the model of *k* vs. equilibrium length *l.*

Hypothesis analysis:
It was predicted as the equilibrium length *l* of the cord increased, *k* should decrease. The relationship of *l* and *k* was found to be *k* = 2.414(1/*l*) - 0.134. As *l* increases, *k* decreases, so the hypothesis was correct.

CONCLUSION:

This experiment modelled an elastic cord as an ideal spring in order to investigate *k*, the cord's "spring constant." This is essential for understanding inherent characteristics of the cord, which can be used to predict how the cord will react to an attached, falling mass in the Bungee Challenge. However, *k* changes as equilibrium length *l* changes. This experiment set out to determine the relationship between *k* and *l* such that given a certain equilibrium length *l, k* could be predicted – and therefore Hooke's Law could be utilized in the Bungee Challenge. The relationship found through this experiment was the equation *k* = 2.414(1/*l*) - 0.134.

Implications of these conclusions:
Because this model exists, it is theoretically possible to predict *k* and therefore model the elastic cord as an ideal spring at any given length. The uncertainty of the coefficient on the term 1/*l* was 3%. This provides confidence that, at least within the realm of the masses used to garner *k* values, the mathematical model *k* = 2.414(1/*l*) - 0.134 is relatively appropriate to predict *k*, the spring constant of the elastic cord modelled as an ideal spring, given *l.* However, this model has its short-comings: it was built only using a small range of masses and equilibrium lengths, so extrapolating these predictions to much heavier masses or longer lengths could prove to be inaccurate.